An improved Interior-Point Algorithm for Large-Scale Shakedown Analysis

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Schematic illustration of different material behaviors under varying thermo-mechanical loading

- purely elastic
- instantaneous collapse
Schematic illustration of different material behaviors under varying thermo-mechanical loading

- **Ratcheting**
- **Alternating plasticity**
- **Shakedown**
Shakedown analysis in industrial applications:

As an example*: Flanged-pipe under variable internal pressure and axial force

* S. Mouhtamid: Anwendung direkter Methoden zur industriellen Berechnung von Grenzlasten mechanischer Komponenten
PhD thesis, IAM, RWTH Aachen, Germany, 2007
INTRODUCTION
LOWER-BOUND SHAKEDOWN ANALYSIS
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OUTLINE OF THE ALGORITHM
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LOWER-BOUND SHAKE-DOWN ANALYSIS

Statical shakedown theorem by Melan*:

If there exists a loading factor $\alpha > 1$ and a time-independent residual stress field $\bar{\sigma}$ such that the yield condition $F \leq 0$ is satisfied for all loads contained within the loading domain $\Omega$ at any time $t$ and at all points $x \in V$ in the volume $V$ of the considered structure, then the system will shake down.

$$F\left(\alpha \sigma^E(x,t) + \bar{\rho}(x), \sigma_y(x)\right) \leq 0$$

where:

$\sigma_y(x)$: yield stress

$\sigma^E(x,t)$: elastic reference stress

Mathematical formulation as an optimization problem:

\[
\max \alpha \\
\sum_{r=1}^{NG} C_r \cdot \bar{\rho}_r = 0 \\
\forall r \in [1, NG], \forall j \in [1, NC]: \\
F(\alpha \sigma_{r}^{E,j} + \bar{\rho}_r, \sigma_{y,r}) \leq 0
\]

where:
- \( r \): considered Gaussian point
- \( NG \): total number of Gaussian points
- \( j \): considered corner of the loading domain \( \Omega \)
- \( NC \): total number of corners of the loading domain
- \( C_r \): equilibrium matrixes which guarantee that \( \bar{\rho}_r \) is self-equilibrated

\( \bar{\rho}_r \): affine linear equality constraints
\( F(\cdot) \): nonlinear, convex inequality constraints
SOLUTION BY INTERIOR-POINT METHOD

\[ \min f(x) = -\alpha \]
\[ A \cdot x = 0 \]
\[ c(x) \geq 0 \]
\[ x \in \mathbb{R}^n \]

introduce slack variables \( w \) and split variables \( y \) and \( z \)

\[ \min f(x) \]
\[ A \cdot x = 0 \]
\[ c(x) - w = 0 \]
\[ x - y + z = 0 \]
\[ w \geq 0, y \geq 0, z \geq 0 \]

introduce barrier parameter \( \mu \)

\[ \min f_\mu(x, y, z, w) \]
\[ A \cdot x = 0 \]
\[ c(x) - w = 0 \]
\[ x - y + z = 0 \]
\[ w > 0, y > 0, z > 0 \]

where:

\[ f_\mu(x) = f(x) - \mu \left[ \sum \log w_j + \sum \log y_i + \sum \log z_i \right] \]

v. Mises yield criterion:

\[ c_{r,j}(x) = 2\sigma_{Y,r}^2 - \|u_r - \alpha a_r^j\|_2^2 \geq 0 \]

solution vector:

\[ x = (u_r, v, \alpha)^T \in \mathbb{R}^n \]
\[ n = 6 \times NG + 1 \]
Karush-Kuhn-Tucker (KKT) conditions:

solution is optimal if the Lagrangian $L$ of the problem possesses a saddle point (necessary and sufficient condition for convex problems)

Lagrangian:  $L = f(x, y, z, w) - \lambda_E \cdot (A \cdot x) - \lambda_I \cdot (c(x) - w) - s \cdot (x - y + z)$

with Lagrange multipliers: $\lambda_E \in \mathbb{R}^{m_E}, \lambda_I \in \mathbb{R}^{m_I}_+, s \in \mathbb{R}^n_+$

Saddle point condition: $\nabla L = 0 \text{ in each direction}$

System of nonlinear equations approximately solved by Newton’s method
Newton’s method:

New variables $\Pi_{k+1}$ of the subsequent iteration step $k+1$ are computed from the variables $\Pi_k$ of the previous one $k$:

$$\Pi_{k+1} = \Pi_k + \gamma_k \cdot \Delta \Pi_k$$

Step values $\Delta \Pi_k$ are the solution of the linearized system:

$$J(\Pi_k) \cdot \Delta \Pi_k = -\nabla L(\Pi_k)$$

where: $J(\Pi_k)$: Jacobian of $\nabla L(\Pi)$

$\gamma_k$: diagonal matrix of damping factors
SOLUTION BY INTERIOR-POINT METHOD

Reduced KKT-system:

\[
\begin{pmatrix}
-(Q + E_1) & A^T & C^T \\
A & 0 & 0 \\
C & 0 & E_2
\end{pmatrix}
\begin{pmatrix}
\Delta x \\
\Delta \lambda_E \\
\Delta \lambda_I
\end{pmatrix}
= 
\begin{pmatrix}
\nabla_x f(x) - A^T \cdot \lambda_E - C^T \cdot \lambda_I - s + E_1 \cdot b_1 \\
-A \cdot x \\
-c(x) + \mu A_I^{-1} \cdot e
\end{pmatrix}
\]

where:
\[
Q = \nabla^2_x L = -\sum_{k=1}^{m_I}(\nabla^2_x c_k(x))_I \lambda_{I,k}
\]
\[
E_1 = \left(S^{-1} \cdot Y + R^{-1} \cdot Z\right)^{-1}
\]
\[
E_2 = W \cdot A_I^{-1}
\]
\[
C = c(x) \nabla_x
\]
\[
b_1 = x + z + \mu \left(R^{-1} - S^{-1}\right) \cdot e + R^{-1} \cdot S \cdot z
\]
\[
e = (1 \cdots 1)^T \text{ in proper dimension}
\]

problem dimension:
\[
(6NG + 1) + m_E + m_I
\]
due to zero-block on diagonal regularization necessary!
Improvement by condensation of KKT-system:

Improved formulation leads to specific structure:

\[
H = Q + E_1 + C^T \cdot E_2^{-1} \cdot C = \begin{pmatrix}
H_u & 0 & h \\
0 & H_v & 0 \\
h^T & 0 & H_\alpha
\end{pmatrix}
\]

and

\[
A = \begin{pmatrix}
A_u & A_v & a_\alpha
\end{pmatrix}
\]

Condensation of KKT-system:

\[
\begin{pmatrix}
-H_u & -h & A_u^T \\
-h^T & -H_\alpha & a_\alpha^T \\
A_u & a_\alpha & A_v \cdot H_v^{-1} \cdot A_v^T
\end{pmatrix}
\begin{pmatrix}
\Delta u \\
\Delta \alpha \\
\Delta \lambda_E
\end{pmatrix}
= rhs
\]

\text{problem dimension:}
\( (5 \, NG + 1) + m_E \)

no regularization necessary!
Damping of the Newton step:

Full Newton step $\Pi_{k+1} = \Pi_k + \Delta \Pi_k$ may happen to be too large.

Newton step is damped with damping factors $\gamma_k$ in order to ensure that:

- all slack and split variables and the Lagrange multipliers remain positive

- $\Pi_{k+1}$ is closer to the solution than $\Pi_k$
  i.e. the infeasibility of constraints and the value of the objective function are decreasing

\textit{Linesearch}
SOLUTION BY INTERIOR-POINT METHOD

Damping with Linesearch:

Merit function: measure of objective function and infeasibilities

\[ \phi_{\mu \nu} = f_{\mu}(x, y, z, w) + \frac{\nu}{2} \left\| \begin{pmatrix} A \cdot x \\ c(x) - w \\ x - y + z \end{pmatrix} \right\|^2 + (A \cdot x) \cdot \lambda_E \]

where: \( \nu \) penalty parameter (updated each iteration step if necessary)

Armijo condition guarantees descent direction in objective and infeasibilities:

\[ \phi_{\mu \nu}(\Pi_k + \theta \Delta \Pi_k) - \phi_{\mu \nu}(\Pi_k) \leq \beta \theta \phi'(\Pi_k; \Delta \Pi_k) \]

Newton step is damped such that Armijo condition is satisfied
**SOLUTION BY INTERIOR-POINT METHOD**

**Initialization:**
- Elastic stresses
- C-matrix
- Transformation

**Solve KKT**

**Update variables**

**Done**

**Update barrier**

**Damp with Non-negativity**

**Damp with Linesearch**

**Update KKT**

**Break cond. outer iter.**

**Break cond. inner iter.**

**Yes**

**No**
Concluding remarks:

• use of interior-point methods leads to efficient algorithms in shakedown analysis

• improved formulation has been presented for v.Mises yield criterion

• condensation of the KKT-system provides reduction of CPU-time

Perspectives:

• extension to larger classes of materials: kinematical hardening, SMA

• numerical advancements: selective algorithm

• industrial application