Institute of General Mechanics
RWTH Aachen University

A Selective Algorithm for Shakedown Analysis using the Interior-Point Method

J.W. Simon, D. Weichert, M. Kreimeier

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Schematic illustration of different material behaviors under varying thermo-mechanical loading

- **purely elastic**
- **instantaneous collapse**
Schematic illustration of different material behaviors under varying thermo-mechanical loading.

- **Ratcheting**
- **Alternating plasticity**
- **Shakedown**
Statical shakedown theorem by Melan*: 

If there exists a loading factor $\alpha > 1$ and a time-independent residual stress field $\bar{\rho}$ such that the yield condition $F_Y \leq 0$ is satisfied for all loads contained within the loading domain $\Omega$ at any time $t$ and at all points $x$ in the volume of the considered structure, then the system will shake down.

$$F_Y \left( \alpha \sigma^E(x,t) + \bar{\rho}(x), \sigma_Y(x) \right) \leq 0$$

where: $\sigma_Y(x)$: yield stress

$\sigma^E(x,t)$: elastic reference stress

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Extension for limited kinematical hardening:

Decomposition: \( \sigma = \pi + \nu \)

- \( \sigma \): total stresses
- \( \pi \): back-stresses
- \( \nu \): stresses responsible for plastic deformation

Initial yield surface: \( F_Y^0(\nu) = 0 \)

Actual yield surface: \( F_Y(\nu) = 0 \)

Bounding surface: \( F_H(\sigma) = 0 \)
Mathematical formulation of the statical shakedown theorem as an optimization problem:

\[
\max \alpha \\
\sum_{r=1}^{NG} C_r \cdot \bar{\rho}_r = 0 \\
\forall r \in [1, NG], \forall j \in [1, NC]:
\]

\[
F_Y \left( \alpha \sigma_{r,j}^E + \bar{\rho}_r - \bar{\pi}_r; \sigma_{Y,r} \right) \leq 0
\]

\[
F_H \left( \alpha \sigma_{r,j}^E + \bar{\rho}_r; \sigma_{H,r} \right) \leq 0
\]

where:

- \( r \): considered Gaussian point
- \( NG \): total number of Gaussian points
- \( j \): considered corner of the loading domain \( \Omega \)
- \( NC \): total number of corners of the loading domain
- \( C_r \): equilibrium matrixes which guarantee that \( \bar{\rho}_r \) is self-equilibrated
\[ \min f(x) = -\alpha \]
\[ A \cdot x = 0 \]
\[ c(x) \geq 0 \]
\[ x \in \mathbb{R}^n \]

introduce slack variables \( w \) and split variables \( y \) and \( z \)

\[ \min f(x) \]
\[ A \cdot x = 0 \]
\[ c(x) - w = 0 \]
\[ x - y + z = 0 \]
\[ w \geq 0, y \geq 0, z \geq 0 \]

introduce barrier parameter \( \mu \)

\[ \min f_\mu(x, y, z, w) \]
\[ A \cdot x = 0 \]
\[ c(x) - w = 0 \]
\[ x - y + z = 0 \]
\[ w > 0, y > 0, z > 0 \]

where: \( f_\mu(x) = f(x) - \mu \left[ \sum \log w_j + \sum \log y_i + \sum \log z_i \right] \)
Karush-Kuhn-Tucker (KKT) conditions:

solution is optimal if the Lagrangian $L$ of the problem possesses a saddle point (necessary and sufficient condition for convex problems)

Lagrangian: $L = f_{\mu}(x, y, z, w) - (A \cdot x) \cdot \lambda_E - (c(x) - w) \cdot \lambda_I - (x - y + z) \cdot s$

with Lagrange multipliers: $\lambda_E \in \mathbb{R}^{m_E}, \lambda_I \in \mathbb{R}_+^{m_I}, s \in \mathbb{R}_+^n$

Saddle point condition: $\nabla L = 0 \quad in each direction$

System of nonlinear equations approximately solved by Newton’s method
Newton’s method:

New variables $\Pi_{k+1}$ of the subsequent iteration step $k+1$ are computed from the variables $\Pi_k$ of the previous one $k$:

$$\Pi_{k+1} = \Pi_k + \Upsilon_k \cdot \Delta\Pi_k$$

Step values $\Delta\Pi_k$ are the solution of the linearized system:

$$J(\Pi_k) \cdot \Delta\Pi_k = -\nabla L(\Pi_k)$$

where: $J(\Pi_k)$: Jacobian of $\nabla L(\Pi_k)$

$\Upsilon_k$: diagonal matrix of damping factors
Initialization:
- elastic stresses
- C-matrix
- transformation

Solve KKT

Update barrier
 Break cond. outer iter.

Update KKT

Damp with Non-negativity

Update variables
 Break cond. inner iter.

Damp with Linesearch

Done
SELECTIVE ALGORITHM

mathematical procedure:
solution of optimization problem

evolution of active zones

mechanical procedure:
stresses due to increase of loading factor

SELECTIVE ALGORITHM
Procedure of Selective Algorithm:

• Start the iteration of the original problem until a prescribed limit

• Determine the plastically critical zones from the intermediate results

  Gaussian Points (GP) are set active if the equivalent stress in the GP is larger or equal to the yield stress multiplied by the prescribed factor $\beta$

  $$\sigma_{eq} \geq \beta \sigma_y \quad \Rightarrow \quad GP \text{ is active}$$

  Surrounding elements are set active for consistency

• Solve the reduced system composed of the critical zones until convergence

• Introduce the solution of the substructure as starting point of the original problem
SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

Square plate with a circular hole:

\( P_x \) and \( P_y \) vary independently

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ( L ) in [mm]</td>
<td>100</td>
</tr>
<tr>
<td>Thickness ( t ) in [mm]</td>
<td>2</td>
</tr>
<tr>
<td>Diameter ( D ) in [mm]</td>
<td>20</td>
</tr>
<tr>
<td>Young’s modulus in [MPa]</td>
<td>2.1\times10^5</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.3</td>
</tr>
<tr>
<td>Yield stress in [MPa]</td>
<td>200</td>
</tr>
</tbody>
</table>
SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

FE-mesh and relevant numbers of optimization problem:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Elements $NE$</td>
<td>400</td>
</tr>
<tr>
<td>Gaussian points $NG$</td>
<td>3 200</td>
</tr>
<tr>
<td>Corners $NC$</td>
<td>4</td>
</tr>
<tr>
<td>Variables</td>
<td>67 201</td>
</tr>
<tr>
<td>Equality constraints $m_E$</td>
<td>50 646</td>
</tr>
<tr>
<td>Inequality constraints $m_I$</td>
<td>12 800</td>
</tr>
</tbody>
</table>

Element-type: solid, 8 nodes per element
solid45 in ANSYS
Results of the entire system:

Number of iterations: 335
Running time: 232 s

working station:
Sun W 1100z
CPU 2.4GHz
RAM 5120 MB

* S. Mouhtamid: Anwendung direkter Methoden zur industriellen Berechnung von Grenzlasten mechanischer Komponenten
PhD thesis, IAM, RWTH Aachen University, Germany, 2007
SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

Active elements: Condition Factor 0.8, Iteration 270/350

Lochscheibe 400, degree 45, Active Zones, Iteration 270
Active elements: Condition Factor 0.8, Iteration 280/350
SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

Active elements: Condition Factor 0.8, Iteration 290/350

Lochscheibe 400, degree 45, Active Zones, Iteration 290
SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

Active elements: Condition Factor 0.8, Iteration 300/350

Lochscheibe 400, degree 45, Active Zones, Iteration 300
SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

Active elements: Condition Factor 0.8, Iteration 310/350

Lochscheibe 400, degree 45, Active Zones, Iteration 310
SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

Active elements: Condition Factor 0.8, Iteration 320/350

Lochscheibe 400, degree 45, Active Zones, Iteration 320
Active elements: Condition Factor 0.8, Iteration 330/350
SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

Active elements: Condition Factor 0.8, Iteration 340/350
SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

Active elements: Condition Factor 0.8, Iteration 350/350

Lochscheibe 400, degree 45, Active Zones, Iteration 350
Manually selected active zone:
### SELECTIVE ALGORITHM – Numerical Example: Square plate with hole

Convergence of the reduced system with manually selected zones

<table>
<thead>
<tr>
<th>nb rings</th>
<th>nb active GP</th>
<th>alpha</th>
<th>running time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>800</td>
<td>1,5607 *</td>
<td>55</td>
</tr>
<tr>
<td>6</td>
<td>960</td>
<td>1,6376 *</td>
<td>62</td>
</tr>
<tr>
<td>7</td>
<td>1120</td>
<td>1,6780</td>
<td>44</td>
</tr>
<tr>
<td>8</td>
<td>1280</td>
<td>1,6764</td>
<td>44</td>
</tr>
<tr>
<td>9</td>
<td>1440</td>
<td>1,6777</td>
<td>52</td>
</tr>
<tr>
<td>10</td>
<td>1600</td>
<td>1,6772</td>
<td>58</td>
</tr>
<tr>
<td>11</td>
<td>1760</td>
<td>1,6775</td>
<td>67</td>
</tr>
<tr>
<td>12</td>
<td>1920</td>
<td>1,6773</td>
<td>87</td>
</tr>
<tr>
<td>entire system</td>
<td>3200 (all)</td>
<td>1,6777</td>
<td>232</td>
</tr>
</tbody>
</table>

* with softening of the convergence criteria
**Material**: steel, all material parameters are considered as temperature-independent

**Loading**: internal pressure, internal temperature

<table>
<thead>
<tr>
<th></th>
<th>Pipe</th>
<th>Nozzle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length [mm]</td>
<td>600.00</td>
<td>157.15</td>
</tr>
<tr>
<td>Thickness [mm]</td>
<td>3.6</td>
<td>2.6</td>
</tr>
<tr>
<td>Inner radius [mm]</td>
<td>53.55</td>
<td>18.60</td>
</tr>
</tbody>
</table>
Element-type: solid, 8 nodes per element

solid45 (structural), solid70 (thermal) in ANSYS

Number of elements: 510
Number of nodes: 1136

Boundary conditions:
Left end of pipe is clamped
Right end of pipe is fixed in longitudinal direction
Nozzle is assumed closed without restrictions on displacements
Equivalent von Mises stresses due to internal pressure:
Equivalent von Mises stresses due to temperature:
Results of the entire system:

Number of iterations: 280
Running time: 302 s

working station:
Sun W 1100z
CPU 2,4GHz
RAM 5120 MB
SELECTIVE ALGORITHM – Numerical Example: Pipe-junction

Active elements: Condition Factor 0.6, Iteration 200/280
SELECTIVE ALGORITHM – Numerical Example: Pipe-junction

Active elements: Condition Factor 0.6, Iteration 220/280
SELECTIVE ALGORITHM – Numerical Example: Pipe-junction

Active elements: Condition Factor 0.6, Iteration 240/280
Active elements: Condition Factor 0.6, Iteration 260/280
SELECTIVE ALGORITHM – Numerical Example: Pipe-junction

Active elements: Condition Factor 0.6, Iteration 280/280
Concluding remarks:

- coherence between the mechanical problem and its mathematical solution procedure is motivation for selective algorithm
- evolution of active zones works very good
- running time can be drastically reduced

Perspectives:

- refine starting-point strategy
- improve adjustment of the variables of inactive elements
- industrial application