Interior-Point Method for the Computation of Shakedown Loads for Engineering Systems

J.W. Simon, D. Weichert

Schematic illustration of different material behaviors under varying thermo-mechanical loading

- purely elastic
- instantaneous collapse
Schematic illustration of different material behaviors under varying thermo-mechanical loading.

- **Ratcheting**
- **Alternating plasticity**
- **Shakedown**
CONTENTS

INTRODUCTION
LOWER-BOUND SHAKEDOWN ANALYSIS
SOLUTION BY INTERIOR-POINT METHOD
NUMERICAL EXAMPLES
CONCLUSIONS
LOWER-BOUND SHAKEDOWN ANALYSIS

Statistical shakedown theorem by Melan*

If there exists a loading factor $\alpha > 1$ and a time-independent residual stress field $\bar{\rho}$ such that the yield condition $F \leq 0$ is satisfied for all loads contained within the loading domain $\Omega$ at any time $t$ and at all points $x \in V$ in the volume $V$ of the considered structure, then the system will shake down.

\[
F\left(\alpha \sigma^E(x,t) + \bar{\rho}(x), \sigma_Y(x)\right) \leq 0
\]

where:  
$\sigma_Y(x)$: yield stress  
$\sigma^E(x,t)$: elastic reference stress

Mathematical formulation as an optimization problem:

\[
\max \alpha \quad \text{linear objective function (convex)}
\]

\[
\sum_{r=1}^{NG} C_r \cdot \bar{\rho}_r = 0 \quad \text{affine linear equality constraints}
\]

\[
\forall r \in [1, NG], \forall j \in [1, NC]:
\]

\[
F \left( \alpha \sigma_{r,j}^E + \bar{\rho}_r, \sigma_{Y,r} \right) \leq 0 \quad \text{nonlinear, convex inequality constraints}
\]

where:

- \( r \): considered Gaussian point
- \( NG \): total number of Gaussian points
- \( j \): considered corner of the loading domain \( \Omega \)
- \( NC \): total number of corners of the loading domain
- \( C_r \): equilibrium matrixes which guarantee that \( \bar{\rho}_r \) is self-equilibrated
SOLUTION BY INTERIOR-POINT METHOD

\[ \min f(x) = -\alpha \]

\[ A \cdot x = 0 \]

\[ c(x) \geq 0 \]

\[ x \in \mathbb{R}^n \]

introduce slack variables \( w \) and split variables \( y \) and \( z \)

\[ \min f(x) \]

\[ A \cdot x = 0 \]

\[ c(x) - w = 0 \]

\[ x - y + z = 0 \]

\[ w \geq 0, y \geq 0, z \geq 0 \]

introduce barrier parameter \( \mu \)

\[ \min f_\mu(x, y, z, w) \]

\[ A \cdot x = 0 \]

\[ c(x) - w = 0 \]

\[ x - y + z = 0 \]

\[ w > 0, y > 0, z > 0 \]

where: \[ f_\mu(x) = f(x) - \mu \left[ \sum \log w_j + \sum \log y_i + \sum \log z_i \right] \]

v. Mises yield criterion:

\[ c_{r,i}(x) = 2\sigma_{Y,r}^2 - \|u_r - \alpha a_r\|^2 \geq 0 \]

solution vector:

\[ x = (u_r \ v \ \alpha)^T \in \mathbb{R}^n \]

\[ n = 6*NG + 1 \]
Karush-Kuhn-Tucker (KKT) conditions:

solution is optimal if the Lagrangian $L$ of the problem possesses a saddle point
(necessary and sufficient condition for convex problems)

Lagrangian: \[ L = f(x, y, z, w) - \lambda_E \cdot (A \cdot x) - \lambda_I \cdot (c(x) - w) - s \cdot (x - y + z) \]

with Lagrange multipliers: $\lambda_E \in \mathbb{R}^{m_E}, \lambda_I \in \mathbb{R}^{m_I}, s \in \mathbb{R}^n$

Saddle point condition: $\nabla L = 0 \quad in each direction$

System of nonlinear equations approximately solved by Newton’s method
Newton’s method:

New variables $\Pi_{k+1}$ of the subsequent iteration step $k+1$ are computed from the variables $\Pi_k$ of the previous one $k$:

$$\Pi_{k+1} = \Pi_k + \Upsilon_k \cdot \Delta \Pi_k$$

Step values $\Delta \Pi_k$ are the solution of the linearized system:

$$J(\Pi_k) \cdot \Delta \Pi_k = -\nabla L(\Pi_k)$$

where:

$J(\Pi_k)$: Jacobian of $\nabla L(\Pi)$

$\Upsilon_k$: diagonal matrix of damping factors
Reduced KKT-system:

\[
\begin{pmatrix}
-Q - E_1 & A^T & C^T \\
A & 0 & 0 \\
C & 0 & E_2
\end{pmatrix}
\begin{pmatrix}
\Delta x \\
\Delta \lambda_E \\
\Delta \lambda_I
\end{pmatrix}
= \begin{pmatrix}
\nabla_{x,f}(x) - A^T \cdot \lambda_E - C^T \cdot \lambda_I - s + E_1 \cdot b_1 \\
-A \cdot x \\
-c(x) + \mu \Lambda^{-1}_I \cdot e
\end{pmatrix}
\]

where:

\[Q = \nabla_x^2 L = -\sum_{k=1}^{m_i} \left( \nabla_x^2 c_k(x) \right) \lambda_{I,k}\]

\[E_1 = \left( S^{-1} \cdot Y + R^{-1} \cdot Z \right)^{-1}\]

\[E_2 = W \cdot \Lambda^{-1}_I\]

\[C = c(x) \nabla_x \]

\[b_1 = x + z + \mu \left( R^{-1} - S^{-1} \right) \cdot e + R^{-1} \cdot S \cdot z\]

\[e = (1 \quad \cdots \quad 1)^T \quad \text{in proper dimension}\]

due to zero-block on diagonal regularization necessary!
Improvement by condensation of KKT-system:

Improved formulation leads to specific structure:

\[
\begin{bmatrix}
H_u & 0 & h \\
0 & H_v & 0 \\
h^T & 0 & H_\alpha
\end{bmatrix}
\]

and

\[
A = \begin{bmatrix} A_u & A_v & a_\alpha \end{bmatrix}
\]

Condensation of KKT-system:

\[
\begin{pmatrix}
-H_u & -h & A_u^T \\
-h^T & -H_\alpha & a_\alpha^T \\
A_u & a_\alpha & A_v \cdot H_v^{-1} \cdot A_v^T
\end{pmatrix}
\begin{pmatrix}
\Delta u \\
\Delta \alpha \\
\Delta \lambda_E
\end{pmatrix}
= \text{rhs}
\]

Problem dimension:
\[(5 \, NG + 1) + m_E\]

No regularization necessary!
SOLUTION BY INTERIOR-POINT METHOD

**Initialization:**
- elastic stresses
- C-matrix
- transformation

**Update:**
- variables
- KKT
- barrier

**Break conditions:**
- outer iter.
- inner iter.

**Termination:**
- YES
- NO
NUMERICAL EXAMPLES

Square plate with a circular hole:

$P_x$ and $P_y$ vary independently

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length $L$ in [mm]</td>
<td>100</td>
</tr>
<tr>
<td>Thickness $t$ in [mm]</td>
<td>2</td>
</tr>
<tr>
<td>Diameter $D$ in [mm]</td>
<td>20</td>
</tr>
<tr>
<td>Young’s modulus in [MPa]</td>
<td>$2.1 \times 10^6$</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.3</td>
</tr>
<tr>
<td>Yield stress in [MPa]</td>
<td>20</td>
</tr>
</tbody>
</table>
NUMERICAL EXAMPLES

FE-mesh and relevant numbers of optimization problem:

<table>
<thead>
<tr>
<th>Elements $NE$</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian points $NG$</td>
<td>3 200</td>
</tr>
<tr>
<td>Corners $NC$</td>
<td>4</td>
</tr>
<tr>
<td>Variables</td>
<td>67 201</td>
</tr>
<tr>
<td>Equality constraints $m_E$</td>
<td>50 646</td>
</tr>
<tr>
<td>Inequality constraints $m_I$</td>
<td>12 800</td>
</tr>
</tbody>
</table>
Results:

Reference value: 0.4666
Present result: 0.4862
Relative error: 4%
Reduction of CPU: 15%
NUMERICAL EXAMPLES

Square plate with a circular hole: Additional temperature load

All three loads $P_x, P_y$ and $T$ vary independently:

- $0 \leq P_x \leq \mu_x P_0$
- $0 \leq P_y \leq \mu_y P_0$
- $0 \leq T \leq \mu_T T_0$

All material parameters are considered as temperature-independent.
NUMERICAL EXAMPLES

Result in three-dimensional loading space:
Concluding remarks:

- Use of interior-point methods leads to efficient algorithms in shakedown analysis.
- Improved formulation has been presented for v. Mises yield criterion.
- Condensation of the KKT-system provides reduction of CPU-time (much higher reduction for more complex structures).
- Method allows calculation of systems with multidimensional loading.

Perspectives:

- Extension to larger classes of materials: kinematical hardening, SMA.
- Numerical advancements: selective algorithm.
- Industrial application.