Numerical Modelling of the Hysteretic Behaviour of Piezoactuated Structures

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Motivation

Aim: Development of a method to determine the piezoelectrically induced deformation in a mechanical structure.
Contents

- Introduction
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Experimental Device

- Actuator: PZT-ceramic PIC151 75x25x0.25 $mm^3$
- Substructure: spring band steel 0.5 $mm$ thickness
- Bonding: electroconductive 2-component epoxy 0.05 $mm$ thickness
Field Equations

$\text{div} \mathbf{T} + \mathbf{f}^* = 0$

$\text{div} \mathbf{D} = 0$

with

$\mathbf{S} = \frac{1}{2} \left( \text{grad} \tilde{\mathbf{u}} + \left( \text{grad} \tilde{\mathbf{u}} \right)^T \right)$

$\mathbf{E} = -\text{grad} \varphi$

$\mathbf{T} = \mathbf{T}^T$

$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

$\mathbf{T}$  stress

$\mathbf{f}^*$  body force

$\mathbf{D}$  electric displacement

$\mathbf{S}$  strain

$\tilde{\mathbf{u}}$  displacement

$\mathbf{E}$  electric field

$\varphi$  electric potential

$\mathbf{P}$  polarisation
Kirchhoff-Love Plate

Assumption: pure bending due to piezo-induced moments

\[ m_{xx} = \frac{E_p}{1 - \nu^2} \left( S_{xp} + \nu S_{yp} \right) t_p (t_s + t_b + t_p/2 - z_s) \]

\[ m_{yy} = \frac{E_p}{1 - \nu^2} \left( S_{yp} + \nu S_{xp} \right) t_p (t_s + t_b + t_p/2 - z_s) \]

piezo-induced strain
Kirchhoff-Love Plate

\[
\frac{\partial^2 w}{\partial x^2} = -\frac{1}{(1 - \nu^2)N}(m_{xx} - \nu m_{yy}) \\
\frac{\partial^2 w}{\partial y^2} = -\frac{1}{(1 - \nu^2)N}(m_{yy} - \nu m_{xx}) \\
S_x = -\frac{\partial^2 w}{\partial x^2} z \\
S_y = -\frac{\partial^2 w}{\partial y^2} z
\]

\(N\) bending stiffness
\(\nu\) Poisson’s ratio
\(w\) z-displacement

The stress in the actuator midplane is taken into account for the coupled solution.
Constitutive Model

Based on the model introduced by Kamlah (2000).

\[
\begin{align*}
\{ S, \bar{P} \} & \quad \text{constitutive model} \quad \{ T, \bar{E} \}
\end{align*}
\]

Basic idea: decomposition into reversible and irreversible parts

\[
\begin{align*}
S &= S^r + S^i \\
\bar{P} &= \bar{P}^r + \bar{P}^i
\end{align*}
\]
Reversible Parts

\[ S^r = c^{-1} : T + d^T \cdot \vec{E} \]
\[ \vec{P}^r = d : T + \epsilon \cdot \vec{E} \]

Material constants \(c^{-1}\) and \(\epsilon \Rightarrow\) isotropic

Piezoelectric coefficient tensor \(d\) aligned with polarisation direction

no polarisation \(\Rightarrow\) no piezoelectricity

\[
d = \left( \delta \frac{||\vec{E}||}{E_c} + \gamma \right) \left( d||\vec{P}^i|| \frac{||\vec{P}^i||}{P_{sat}} \vec{e}_{Pi} \otimes \vec{e}_{Pi} \otimes \vec{e}_{Pi} + d_{1} \frac{||\vec{P}^i||}{P_{sat}} \vec{e}_{Pi} \otimes (I - \vec{e}_{Pi} \otimes \vec{e}_{Pi}) + d_{2} \frac{||\vec{P}^i||}{P_{sat}} \left( (I \otimes \vec{e}_{Pi}) + (I \otimes \vec{e}_{Pi})^{T23} - 2 \vec{e}_{Pi} \otimes \vec{e}_{Pi} \otimes \vec{e}_{Pi} \right) \right)
\]
Irreversible Strain

\[ S^i = S^{ie} + S^{im} \]

Decomposition into electrically \((S^{ie})\) and mechanically \((S^{im})\) induced parts.

\[ S^{ie}(\vec{P}^i) = \frac{3}{2} S_{sat} \frac{||\vec{P}^i||}{P_{sat}} \left( \bar{\epsilon}_{Pi} \otimes \bar{\epsilon}_{Pi} - \frac{1}{3} I \right) \]

\[ S = S^r + S^i = c^{-1} : T + \text{piezo-induced strain} \]

\[ \vec{P} = \vec{P}^r + \vec{P}^i \]

\[ \Rightarrow \text{Evolution equations for } S^{im} \text{ and } \vec{P}^i \text{ are needed.} \]
Evolution Equations for $\vec{P}^i$

Ferroelectric yield and saturation conditions:

\[
\begin{align*}
    f^e(\vec{E}, \vec{P}^i) &= ||\vec{E} - c^e \vec{P}^i|| - E_c \\
    h^e(T, \vec{E}, \vec{P}^i) &= ||\vec{P}^i|| - \hat{P}_{sat}(T, \vec{E}, \vec{P}^i) \quad \text{higher priority!} \\
    \hat{P}_{sat}(T, \vec{E}, \vec{P}^i) &= P_{sat} e^{-\frac{1}{m} \left\langle \frac{3\rho}{2} \vec{e}_{pi} \cdot T^D \cdot \vec{e}_{pi} - \hat{T}_c(\vec{E}, \vec{P}^i) \right\rangle} \\
    c^e &= a + b e^{||P^i||_i}
\end{align*}
\]

(Grünbichler et al. 2008)
Evolution Equations for $S^{im}$

Ferroelastic yield and saturation conditions:

\[ f^m(\vec{E}, \vec{P}^i, T, S^{im}) = \sqrt{\frac{3}{2}}||T - c^F S^{im}D|| - \hat{T}_c(\vec{E}, \vec{P}^i) \]

\[ h^m(\vec{P}^i, S^{im}) = \sqrt{\frac{2}{3}}||S^{im}|| - \left( S_{sat} - \sqrt{\frac{2}{3}}||S^{ie}|| \right) \]

\[ \hat{T}_c(\vec{E}, \vec{P}^i) = \left\langle \sigma_c + n \frac{\vec{E}}{E_c} \cdot \vec{e}_{P^i} \right\rangle \]

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    xlabel={strain \( \epsilon \) [%]},
    ylabel={stress \( \sigma \) [N/mm²]},
    xmin=-0.4, xmax=0.4,
    ymin=-100, ymax=100,
    xtick={-0.4,-0.3,-0.2,-0.1,0,0.1,0.2,0.3,0.4},
    ytick={-100,-80,-60,-40,-20,0,20,40,60,80,100},
    grid=both,
    axis lines=middle,
]
\addplot[only marks,mark=+,mark options={black},mark size=1.5] coordinates {
    (-0.4,100) (-0.3,100) (-0.2,100) (-0.1,100) (0,0) (0.1,0) (0.2,100) (0.3,100) (0.4,100)
};
\end{axis}
\end{tikzpicture}
\end{center}
Radial Return Mapping

\[ n+1 \bar{P}_i = n \bar{P}_i + \Delta \bar{P}_f^i + \Delta \bar{P}_h^i \]
\[ n+1 S_{im} = n S_{im} + \Delta S_f^{im} + \Delta S_h^{im} \]

The correctors are determined with the help of scalar parameters in such a way that the flow and saturation conditions are not violated.

\[ \Delta \bar{P}_f^i = \alpha^e \frac{\partial f^e(n+1 \bar{E}, n \bar{P}_i)}{\partial \bar{E}} \]
\[ \Delta S_f^{im} = \alpha^m \frac{\partial f^m(n+1 \bar{T}, n+1 \bar{E}, n \bar{P}_i + \Delta \bar{P}_f^i, n S_{im})}{\partial S_{im}} \]
\[ \Delta \bar{P}_h^i = \beta^e \frac{\partial h^e(n+1 \bar{T}, n+1 \bar{E}, n \bar{P}_i + \Delta \bar{P}_f^i, n S_{im} + \Delta S_f^{im})}{\partial \bar{P}_i} \]
\[ \Delta S_h^{im} = \beta^m(\beta^e) \frac{\partial h^m(n \bar{P}_i + \Delta \bar{P}_f^i + \Delta \bar{P}_h^i, n S_{im} + \Delta S_f^{im})}{\partial \bar{E}} \]
Actuator Characterisation by Michelson Interferometry

- fixed mirror
- moveable mirror
- lens
- photodiode
- semipermeable mirror
- interference pattern
- laser
- piezoelectric actuator
- guideway
- clamping slide
Results of the Free Actuator

![Graph showing the relationship between Electric Field $E_z$ [kV/mm] and Strain S [-]. The graph compares the model and experiment results.]
Results of the Active Structure

Tip Displacement $w$ [mm] vs. Electric Field $E_z$ [kV/mm]

- Model (decoupled)
- Model (coupled)
- Experiment
Summary / Outlook

- basic behaviour of active structure can be predicted
- refinement of constitutive model
- FEM implementation
Thanks for your attention!