INTRODUCTION

PROBLEM
A mechanical structure or structural element made of composite materials operates beyond the elastic limit:

- Determination of **material properties**
- **Variable loads** with unknown evolution in time

METHOD

Direct methods combined with **homogenization technique**

- Direct methods give information on serviceability without calculating the evolution of mechanical field quantities.
  - Instantaneous collapse → **Limit Analysis**
  - Failure under variable loads → **Shakedown Analysis**
- Prediction of the global material properties by using homogenization theory
CONTENTS

- **Basic concepts**
  - Static shakedown theorem

- **Direct methods applied to composites**
  - Elements of homogenization theory
  - Boundary conditions
  - Consideration of kinematic hardening

- **Failure criterion of composites**
  - Loci of yield strength fitting

- **Numerical examples**
  - Periodic fiber reinforced metal matrix composites
  - Porous material

- **Conclusions**
BASIC CONCEPTS

Schematic illustration of different material behaviors

- **Purely elastic**: $\varepsilon^p(x, t) = 0$
- **Shakedown**: $\lim_{t \to \infty} \dot{\varepsilon}^p_{ij} = 0$
- **Low-cycle fatigue**: $\Delta \varepsilon^p(x) = \int_0^T \dot{\varepsilon}^p_{ij}(x, t) \, dt = 0$
- **Ratcheting**: $\Delta \varepsilon^p(x) = \int_0^T \dot{\varepsilon}^p_{ij}(x, t) \, dt \neq 0$
Definition of load domain*

A finite number $n$ of types of loads:

$$P(x, t) = P[\beta_s(t), x] \quad x \in V \text{ or } S_p; \quad s = 1, \ldots, r; \quad \beta_s^- \leq \beta_s(t) \leq \beta_s^+ \quad s = 1, \ldots, n$$

Loading domain $P$ can be described by a $n$-dimensional polyhedron:

$$P(x, t) = \left\{ P|P = \sum_{s=1}^{n} \mu_s(t) P_s^0(x), \quad \mu_s(t) \in [\mu_i^-, \mu_i^+] \right\}$$

Two dimensional loading domain $\mathcal{L}$ and $\alpha\mathcal{L}$

---

**Static shakedown theorem by Melan**

If there exist a loading factor \( \alpha > 1 \) and a time-independent residual stress field \( \overline{\rho}(x) \) whose superposition with elastic stresses \( \sigma^E \) does not exceed the yield condition \( F \leq 0 \) at any time \( t > 0 \) and at all points \( x \in V \) in volume \( V \) of the considered structure,

\[
F \left( \alpha \sigma^E(x, t) + \overline{\rho}(x), \sigma_Y(x) \right) \leq 0
\]

where: \( \sigma_Y(x) \) is yield stress.

\( \sigma^E(x, t) \) is purely elastic stress reference.

then the system will shakedown under arbitrary load paths contained within given load domain \( \mathcal{L} \).

---

Numerical implementation

- Principle of virtual work
  \[ \int_V \{\delta \varepsilon\}^T \{\alpha \sigma^E + \overline{\rho}\} \, dV = \int_{\partial V} \{\delta \mathbf{u}\}^T \{\mathbf{p}^*\} \, dS + \int_V \{\delta \mathbf{u}\}^T \{\mathbf{f}^*\} \, dV \]

- Finite element discretization
  - A purely elastic reference stress \( \sigma^E \) is calculated for each loading vertex by means of conventional FE-analysis.
  - Equilibrium conditions for \( \overline{\rho} \) are satisfied by principle of virtual work *:
    \[ \int_V \{\delta \varepsilon\}^T \{\overline{\rho}\} \, dV = \{\delta \mathbf{u}\} \int_V \mathbf{B}^T \{\overline{\rho}\} \, dV = 0 \]
    \[ \Rightarrow \int_{V_e} \mathbf{B}^T \{\overline{\rho}\} \, dV = \iiint_{-1}^{1} \mathbf{B}^T \{\overline{\rho}\} |\mathbf{J}| \, dr \, ds \, dt = \sum_{j=1}^{NE} |\mathbf{J}| \mathbf{B}^T \{\overline{\rho}\} = 0 \]
    \[ \Rightarrow \sum_{k=1}^{NE} \sum_{j=1}^{NGE} |\mathbf{J}| \mathbf{B}^T \{\overline{\rho}\} = [\mathbf{C}]\{\overline{\rho}\} = 0 \]

\( \sigma^E \) and \([\mathbf{C}]\) are input data for the subsequent SD/LA-module.

BASIC CONCEPTS

Numerical implementation

- **Mathematic formulation of shakedown problem**

\[
\begin{align*}
\max & \quad \alpha \\
\text{s.t.} & \quad \{C\}\{\rho\} = 0 \\
& \quad F[\alpha \sigma_i^F(P_k) + \bar{\rho}_i, \sigma_Y] \leq 0 \\
& \quad i \in [1, NGS], k \in [1, 2^n]
\end{align*}
\]

- **Objective function** → Load factor $\alpha$
- **Variables** → $\alpha$ and residual stress field $\bar{\rho}$
- **Linear equality constraints** → Self-equilibrated condition
- **Nonlinear non-equality constraints** → Yield condition

with $NGS$ is the total number of Gauss Points; $n$ is the number of independent loads; $P_k$ is the load vertex. $k=1$ corresponds to limit analysis; $k=2^n$ corresponds to shakedown analysis.

- **Large-scale optimization**

  - **Algorithm**
    - Augmented Lagrangian method
    - Sequential quadratic programming
    - Interior Point Method
  - **Software Packages**
    - LANCELOT *, ...
    - SNOPT, NPSOL, NLPQL
    - IPDCA**, IPOPT*** ...

  ……


DIRECT METHODS APPLIED TO COMPOSITES

Objective

Micro-/Mesoscopic Level

- Numerical **Model** of limit and shakedown analysis of RVE
- Numerical **Solution** of limit and shakedown problem (FEM & Optimization)

Macroscopic Level

- **Comparison** of loading carrying capacity of composites with different fiber distributions
- **Prediction** of elastic and plastic material properties

Assumptions

- Periodic composites
- At least one ductile phase
Homogenization theory

- Concept of representative volume element (RVE)

\[ \xi = x/\theta, \quad \theta: \text{a small parameter} \]

- Average field quantities *

\[ \Sigma(x) = \frac{1}{V} \int_{V} \sigma(\xi) \, dV = \langle \sigma(\xi) \rangle \]

\[ E(x) = \frac{1}{V} \int_{V} \varepsilon(\xi) \, dV = \langle \varepsilon(\xi) \rangle \]

DIRECT METHODS APPLIED TO COMPOSITES

- Static direct methods for periodic composites *
  \[ \Sigma = \frac{1}{V} \int \left( \alpha \sigma^E + \bar{\rho} \right) dV = \frac{1}{V} \int \alpha \sigma^E dV + \frac{1}{V} \int \bar{\rho} dV \]
  with \[ \frac{1}{V} \int \bar{\rho} dV = 0 \]

Localization problem – Boundary conditions **

- Strain method
  Uniform strain is imposed on \( \partial V \): \[ \mathbf{u} = E \cdot \xi \] on \( \partial V \)

- Stress method
  Uniform stress is imposed on \( \partial V \): \[ \sigma \cdot \mathbf{n} = \Sigma \cdot \mathbf{n} \] on \( \partial V \)

- Periodicity
  Anti-periodicity of stress : \( \sigma \cdot \mathbf{n} \) anti-periodic on \( \partial V \)
  Decomposition of local strain : \( \varepsilon(\mathbf{u}) = E + \varepsilon(\mathbf{u}^{\text{per}}) = E + \varepsilon^{\text{per}} \)
  Average of \( \varepsilon^* \) over the RVE : \[ \langle \varepsilon^{\text{per}} \rangle = 0 \]

**DIRECT METHODS APPLIED TO COMPOSITES**

### Periodic composites – strain method *

\[ \begin{align*}
\mathcal{P}_{\text{strain}} \quad & \begin{cases}
\text{div } \sigma^E = 0 & \text{in } V \\
\sigma^E = \mathbf{d} : (\mathbf{E} + \mathbf{\varepsilon}^\text{per}) & \text{in } V \\
\sigma^E \cdot \mathbf{n} & \text{anti-periodic on } \partial V \\
\mathbf{u}^\text{per} & \text{periodic on } \partial V \\
\langle \mathbf{\varepsilon} \rangle = E
\end{cases}
\end{align*} \]

### Finite element discretization using non-conforming element **

- Material Model: elastic-perfectly plastic
- von Mises yield criterion

\[ \max \alpha \quad \begin{cases}
[C]\{\mathbf{\bar{\rho}}\} = 0 \\
F[\alpha \sigma^E_i (\mathbf{\hat{\rho}}_k) + \mathbf{\bar{\rho}}_i, \sigma_{Yi}] \leq 0 \\
i \in [1, NGS], k = 1, \ldots, 2^n
\end{cases} \]

Nr. Variables: 6NGS+1  
Nr. Equality Constraint: 3NK+9NE  
Nr. Inequality Constraint: NL*NGS  
NL=1, limit analysis;  
NL=2^n, shakedown analysis

---

DIRECT METHODS APPLIED TO COMPOSITES

Consideration of kinematic hardening

- **Unlimited kinematic hardening**
  
  \[
  \sup \{ \alpha \} \text{ s.t. } F[\alpha \sigma^E + \bar{\rho} - \pi, \sigma_Y] \leq 0
  \]

  where: \( \pi \) is back stress.

- **Limited kinematic hardening**
  
  \[
  \sup \{ \alpha \} \text{ s.t. } F[\alpha \sigma^E + \bar{\rho} - \pi, \sigma_Y] \leq 0
  
  F[\alpha \sigma^E + \bar{\rho}, \sigma_U] \leq 0^*
  
  \text{OR } F[\pi, \sigma_U - \sigma_Y] \leq 0^{**}
  \]

  where: \( \sigma_U \) is ultimate stress.

---

FAILURE CRITERION OF COMPOSITES

**Failure**: Every material has certain strength, expressed in terms of stress or strain, beyond which the structures fracture or fail to carry the load.

For heterogeneous material, consisted of two or more than two phases materials, how to determine the *Failure Criterion*?

**Why Need Failure Criterion for Composites?**
- To determine weak and strong directions
- To guide local design of composites
- To guide global design of composites-structures

**Macromechanical Failure Theories in Composites??**
- Maximum stress theory
- Maximum strain theory
- Tsai-Hill theory (Deviatoric strain energy theory)
- Tsai-Wu theory (Interactive tensor polynomial theory)
- ......
- Loci of yield strength fitting based on limit macroscopic stress domain
Hill’s yield criterion in plane stress*

\[ \sigma_1^2 + \sigma_2^2 - \left( \frac{2R}{R+1} \right) \sigma_1 \sigma_2 - X^2 = 0 \]

where \( R = 2 \left( \frac{Z}{X} \right)^2 - 1 \)

with \( X = \frac{1}{\sqrt{F+H}} \) and \( Z = \frac{1}{\sqrt{2F}} \)

\( R \) : is a measure of the plastic anisotropy of a rolled metal

The plastic strain ratio \( R \) is a parameter that indicates the ability of a sheet metal to resist thinning or thickening when subjected to either tensile or compressive forces in the plane of the sheet.

* Hosford, W.F. : Oxford University (1993)
Hill’s yield criterion in plane strain (general stress state)

“y-convention”, i.e. $(Z,Y’,Z’)$:

- Rotate the $x_1y_1z_1$-system about the $z_1$-axis ($Z$) by angle $\Psi$;
- Rotate the current system about the new $y$-axis ($Y’$) by angle $\Theta$;
- Rotate the current system about the new $z$-axis ($Z’$) by angle $\Phi$.

\[
\begin{bmatrix}
  x_2 \\
  y_2 \\
  z_2 \\
\end{bmatrix} = T^{-1} \begin{bmatrix}
  x_1 \\
  y_1 \\
  z_1 \\
\end{bmatrix}
\]

Rotation matrix is:

\[
T_{x_1y_1z_1} = \begin{pmatrix}
  \cos \Psi & -\sin \Psi & 0 \\
  \sin \Psi & \cos \Psi & 0 \\
  0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
  \cos \Theta & 0 & \sin \Theta \\
  0 & 1 & 0 \\
  -\sin \Theta & 0 & \cos \Theta \\
\end{pmatrix} \begin{pmatrix}
  \cos \Phi & -\sin \Phi & 0 \\
  \sin \Phi & \cos \Phi & 0 \\
  0 & 0 & 1 \\
\end{pmatrix}
\]

with

\[
\Psi = \frac{\pi}{4} \quad \Theta = \tan^{-1}(\sqrt{2}) \quad \Phi = -\frac{\pi}{4}
\]
Hill’s yield criterion in plane strain (general stress state)

- Calculation of principle stress

\[ \sigma_{ij} \text{ Dependent on coordinate system} \quad \text{Stress Invariants} \quad \sigma_n \text{ Independent on coordinate system} \]

- Projection in \( \pi \)-plane

\[
F(\sigma_2 - \sigma_3)^2 + G(\sigma_3 - \sigma_1)^2 + H(\sigma_1 - \sigma_2)^2 - 1 = 0
\]

\( F, G, H \) are defined as:

\[
F = \frac{1}{2}\left(\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}\right) ; \quad G = \frac{1}{2}\left(\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}\right) ; \quad H = \frac{1}{2}\left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}\right)
\]

Let:

\[
\begin{align*}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{align*} = T
\begin{align*}
\gamma_1 \\
\gamma_2 \\
\gamma_3
\end{align*}
\]

\[
(F + 1.866H + 0.134G)\gamma_1^2 + (F + 0.134H + 1.866G)\gamma_2^2 + (G - 2F + H)\gamma_1\gamma_2 = 1
\]
Hill’s yield criterion in plane strain (general stress state)

\[(F + 1.866H + 0.134G)\gamma_1^2 + (F + 0.134H + 1.866G)\gamma_2^2 + (G - 2F + H)\gamma_1\gamma_2 = 1\]

For homogenous material, \( X = Y = Z \), i.e. \( F = G = H \):

\[\gamma_1^2 + \gamma_2^2 = C \quad \text{; here: } C = \frac{1}{3F} = \frac{2}{3} \sigma_Y^2\]

For transversely homogenous material, assume \( Y = Z \), i.e. \( G = H \)

\[(F + 2H)\gamma_1^2 + (F + 2H)\gamma_2^2 + (2H - 2F)\gamma_1\gamma_2 = 1\]
NUMERICAL EXAMPLES

Periodic fiber reinforced metal matrix composites

Unidirectional fiber reinforced periodic composites

Representative volume element

Finite element model

Material properties:

<table>
<thead>
<tr>
<th></th>
<th>$E$ (MPa)</th>
<th>$\nu$</th>
<th>$\sigma_Y$ (MPa)</th>
<th>$\sigma_U$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrix (Al)</td>
<td>70e3</td>
<td>0.3</td>
<td>80</td>
<td>120</td>
</tr>
<tr>
<td>Fiber (Al$_2$O$_3$)</td>
<td>370e3</td>
<td>0.3</td>
<td>2000</td>
<td>---</td>
</tr>
</tbody>
</table>

Periodic composites:
Perfect bounding
Different fiber distribution
Different volume fraction (0 ~ 50%)
**NUMERICAL EXAMPLES**

- Homogenized elastic material properties
  - Elastic-perfectly plastic

![Square pattern](image1.png)

![Rotated pattern](image2.png)

![Hexagonal pattern](image3.png)

**Homogenized Poisson Ratio**

![Graph showing homogenized Poisson ratio vs. fiber volume fraction for different patterns](graph1.png)

**Homogenized Young's Modulus**

![Graph showing homogenized Young's modulus vs. fiber volume fraction for different patterns](graph2.png)
NUMERICAL EXAMPLES

- Displacement domain for kinematic hardening

Matrix:
- Elastic-perfectly plastic (SD)
- Unlimited kinematic hardening (SDH-UN)
- Limited kinematic hardening (SDH)

Fiber:
- Square pattern distributed
- Elastic-perfectly plastic
- Volume fraction 40%

Admissible shakedown displacement domain

Admissible limit displacement domain
NUMERICAL EXAMPLES

- Macroscopic stress domain for kinematic hardening

Matrix:
- Elastic-perfectly plastic (SD)
- Unlimited kinematic hardening (SDH-UN)
- Limited kinematic hardening (SDH)

Fiber:
- Square pattern distributed
- Elastic-perfectly plastic
- Volume fraction 40%

Admissible macroscopic shakedown stress domain

Admissible macroscopic limit stress domain
NUMERICAL EXAMPLES

- Prediction of plastic material properties
  - Elastic-perfectly plastic

Major axis: $a=241.83$
Minor axis: $b=67.78$

$X = 296.18 \text{ Mpa} = 3.70 \sigma_m$
$Y = Z = 98.5 \text{ MPa} = 1.23 \sigma_m$
NUMERICAL EXAMPLES

Porous material

Material properties:

<table>
<thead>
<tr>
<th></th>
<th>E (MPa)</th>
<th>$\nu$</th>
<th>$\sigma_Y$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel (Outside)</td>
<td>200e3</td>
<td>0.30</td>
<td>360</td>
</tr>
<tr>
<td>Aluminum (Inside)</td>
<td>72e3</td>
<td>0.33</td>
<td>100</td>
</tr>
</tbody>
</table>

Dimensions of RVE

<table>
<thead>
<tr>
<th>Items</th>
<th>Dimensions (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width</td>
<td>8</td>
</tr>
<tr>
<td>Height</td>
<td>6</td>
</tr>
<tr>
<td>Radius of the hole</td>
<td>1.5</td>
</tr>
<tr>
<td>Thickness of single layer</td>
<td>2</td>
</tr>
<tr>
<td>Thickness of two layers</td>
<td>4</td>
</tr>
</tbody>
</table>
NUMERICAL EXAMPLES

Porous material

- Displacement domain

- Macroscopic stress domain

\[
\begin{align*}
\Sigma_{22}(\text{MPa}) & \quad \Sigma_{11}(\text{MPa}) \\
U_{2}/U_0 & \quad U_1/U_0 \\
U_{20}/U_0 & \quad \Sigma_{11}(\text{MPa}) \\
\end{align*}
\]
CONCLUSIONS

Summary

• Application of direct methods to composites. Three boundary conditions are discussed.

• Prediction of transverse elastic material properties of unidirectional continuous fiber reinforced metal matrix composites based on homogenization theory.

• Consideration of the hardening enlarged the shakedown and limit domain.

• Definition of yield loci for periodic composites and the prediction of plastic material properties by using yield surface fitting.

Perspectives

• Apply direct methods to other composites types and more complex problems.

• Consider thermal loads, as well as the temperature dependent yield strength.

• The presented results for the fiber-reinforced composite are only numerical, and a future effort has to be made in order to compare with experimental results quantitatively.
Thanks for your attention!