Schematic illustration of different material behaviors under varying thermo-mechanical loading

- purely elastic
- instantaneous collapse
Schematic illustration of different material behaviors under varying thermo-mechanical loading.
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LOWER-BOUND SHAKEDOWN ANALYSIS

Statically shakedown theorem by Melan*:

If there exists a loading factor $\alpha > 1$ and a time-independent residual stress field $\overline{\rho}$ such that the yield condition $F_Y \leq 0$ is satisfied for all loads contained within the loading domain $\Omega$ at any time $t$ and at all points $x$ in the volume of the considered structure, then the system will shake down.

$$F_Y\left(\alpha \sigma^E(x,t) + \overline{\rho}(x), \sigma_Y(x)\right) \leq 0$$

where: $\sigma_Y(x)$: yield stress

$\sigma^E(x,t)$: elastic reference stress

Extension for limited kinematical hardening:

Decomposition: \( \sigma = \pi + \nu \)

- \( \sigma \): total stresses
- \( \pi \): back-stresses
- \( \nu \): stresses responsible for plastic deformation

Initial yield surface: \( F_Y^0 (\nu) = 0 \)

Actual yield surface: \( F_Y (\nu) = 0 \)

Bounding surface: \( F_H (\sigma) = 0 \)
Mathematical formulation of the statical shakedown theorem as an optimization problem:

\[
\max \alpha \\
\sum_{r=1}^{NG} \mathbb{C}_r \cdot \bar{\rho}_r = 0 \\
\forall r \in [1, NG], \forall j \in [1, NC]: \\
F_Y \left( \alpha \sigma_r^{E,j} + \bar{\rho}_r - \bar{\pi}_r; \sigma_{Y,r} \right) \leq 0 \\
F_H \left( \alpha \sigma_r^{E,j} + \bar{\rho}_r; \sigma_{H,r} \right) \leq 0
\]

where:
- \( r \): considered Gaussian point
- \( NG \): total number of Gaussian points
- \( j \): considered corner of the loading domain \( \Omega \)
- \( NC \): total number of corners of the loading domain
- \( \mathbb{C}_r \): equilibrium matrixes which guarantee that \( \bar{\rho}_r \) is self-equilibrated
- linear objective function (convex)
- affine linear equality constraints
- nonlinear, convex inequality constraints
Transformation of the problem:

\[
\min f(x) = -\alpha
\]
\[
A \cdot x = 0
\]
\[
c(x) \geq 0
\]
\[
x \in \mathbb{R}^n
\]

\[
\text{introduce slack variables } w \text{ and split variables } y \text{ and } z
\]

\[
\text{introduce barrier parameter } \mu
\]

\[
\min f_\mu(x, y, z, w)
\]
\[
A \cdot x = 0
\]
\[
c(x) - w = 0
\]
\[
x - y + z = 0
\]
\[
w > 0, y > 0, z > 0
\]

where:

\[
f_\mu(x) = f(x) - \mu \left[ \sum \log w_j + \sum \log y_i + \sum \log z_i \right]
\]
Karush-Kuhn-Tucker (KKT) conditions:

Solution is optimal if the Lagrangian $L$ of the problem possesses a saddle point (necessary and sufficient condition for convex problems)

Lagrangian: $L = f_{\mu}(x, y, z, w) - (A \cdot x) \cdot \lambda_E - (c(x) - w) \cdot \lambda_I - (x - y + z) \cdot s$

with Lagrange multipliers: $\lambda_E \in \mathbb{R}^{mE}, \lambda_I \in \mathbb{R}_{+}^{mI}, s \in \mathbb{R}_{+}^{n}$

Saddle point condition: $\nabla L = 0 \quad in \ each \ direction$

System of nonlinear equations approximately solved by Newton’s method
Newton’s method:

New variables $\Pi_{k+1}$ of the subsequent iteration step $k+1$ are computed from the variables $\Pi_k$ of the previous one $k$:

$$\Pi_{k+1} = \Pi_k + \Upsilon_k \cdot \Delta \Pi_k$$

Step values $\Delta \Pi_k$ are the solution of the linearized system:

$$J(\Pi_k) \cdot \Delta \Pi_k = -\nabla L(\Pi_k)$$

where: $J(\Pi_k)$: Jacobian of $\nabla L(\Pi_k)$

$\Upsilon_k$: diagonal matrix of damping factors
**SOLUTION BY INTERIOR-POINT METHOD**

**Initialization:**
- elastic stresses
- C-matrix
- transformation

**Update barrier**
**Update KKT**
**Update variables**

**Break cond. inner iter.**
**Break cond. outer iter.**

**Solve KKT**

**Done**

**Damp with Non-negativity**

**Damp with Linesearch**
Thin pipe under internal pressure and temperature:

$\Delta T$ and $p$ vary independently

all parameters are assumed as temperature-independent

$R/h = 10$
Numerical Example: Thin pipe under thermomechanical loading

FE-mesh and relevant numbers of optimization problem:

Element-type: solid, 8 nodes per element
solid45, solid70 in ANSYS

<table>
<thead>
<tr>
<th>Element</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elements $NE$</td>
<td>600</td>
</tr>
<tr>
<td>Gaussian points $NG$</td>
<td>4800</td>
</tr>
<tr>
<td>Nodes $NK$</td>
<td>984</td>
</tr>
<tr>
<td>Corners $NC$</td>
<td>4</td>
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<tr>
<td>Variables</td>
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<tr>
<td>Equality constraints $m_E$</td>
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<tr>
<td>Inequality constraints $m_I$</td>
<td>38,400</td>
</tr>
</tbody>
</table>
Numerical Example: Thin pipe under thermomechanical loading

Results of shakedown analysis:

Nbr of iterations: 7645
Running time: 2144 s

working station:
Sun W 1100z
CPU 2,4GHz
RAM 5120 MB
Central path of barrier problem:

central path = curve of optimal solutions $\bar{x}(\mu)$ for any barrier parameter $\mu$

The starting point shall be close to the central path.
Choice of solution vector:

The starting point shall be feasible,
the subsidiary conditions shall be satisfied:

\[ A \cdot x_0 = 0 \]
\[ c(x_0) - w_0 = 0 \]
\[ x_0 - y_0 + z_0 = 0 \]

Two possibilities for solution vector:

1) \( x_0 = 0 \) feasible, but not well-centered

2) \( x_0 = x_{\text{elastic}} \) feasible and well-centered

From the subsidiary conditions:
\[ w_0 = c(x_0) \]
\[ z_0 = y_0 - x_0 \]

where: \( y_0 = 10^3 e \) chosen arbitrarily, but not too small
Choice of Lagrange multipliers:

Recall saddle point condition of Lagrangian:

\[ \nabla L = \begin{bmatrix} -\nabla f + A^T \cdot \lambda_{E,0} + C^T \cdot \lambda_{I,0} + s_0 \\ \mu e - Y_0 \cdot S_0 \cdot e \\ \mu e - Z_0 \cdot R_0 \cdot e \\ \mu e - W_0 \cdot \Lambda_{I,0} \cdot e \\ A \cdot x_0 \\ c(x_0) - w_0 \\ x_0 - y_0 + z_0 \\ r_0 + s_0 \end{bmatrix} = 0 \]

Problem: \[ A^T \cdot \lambda_{E,0} = \nabla f - C^T \cdot \lambda_{I,0} - s_0 \]

we choose: \[ \lambda_{E,0} = 10^{-2} e \]
STARTING-POINT – Numerical Example: Square plate with hole

Square plate with a circular hole:

- \( P_x \) and \( P_y \) vary independently
- elastic – perfectly plastic behavior

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Length ( L ) in [mm]</td>
<td>100</td>
</tr>
<tr>
<td>Thickness ( t ) in [mm]</td>
<td>2</td>
</tr>
<tr>
<td>Diameter ( D ) in [mm]</td>
<td>20</td>
</tr>
<tr>
<td>Young’s modulus in [MPa]</td>
<td>( 2.1 \times 10^5 )</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.3</td>
</tr>
<tr>
<td>Yield stress in [MPa]</td>
<td>200</td>
</tr>
</tbody>
</table>
STARTING-POINT – Numerical Example: Square plate with hole

FE-mesh and relevant numbers of optimization problem:

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Elements NE</td>
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<tr>
<td>Gaussian points NG</td>
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<td>Corners NC</td>
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<td>Variables</td>
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<tr>
<td>Equality constraints $m_E$</td>
<td>50646</td>
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<tr>
<td>Inequality constraints $m_I$</td>
<td>12800</td>
</tr>
</tbody>
</table>

Element-type: solid, 8 nodes per element

*solid45 in ANSYS*
Results of the entire system:

working station:
Dell Precision T7500
CPU 2.4 GHz
RAM 12 GB

* S. Mouhtamid: Anwendung direkter Methoden zur industriellen Berechnung von Grenzlasten mechanischer Komponenten
PhD thesis, IAM, RWTH Aachen University, Germany, 2007
Comparison of results with different starting points:

<table>
<thead>
<tr>
<th>φ</th>
<th>nbr of iter</th>
<th>ORIGINAL* time [s]</th>
<th>α</th>
<th>nbr of iter</th>
<th>PRESENT time [s]</th>
<th>α</th>
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</thead>
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<td>249</td>
<td>40,0</td>
<td>1,7626</td>
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<tr>
<td>10°</td>
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<td>55,4</td>
<td>1,6805</td>
<td>195</td>
<td>34,9</td>
<td>1,6805</td>
</tr>
<tr>
<td>20°</td>
<td>349</td>
<td>50,3</td>
<td>1,6538</td>
<td>160</td>
<td>30,2</td>
<td>1,6537</td>
</tr>
<tr>
<td>30°</td>
<td>324</td>
<td>47,5</td>
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<td>159</td>
<td>30,2</td>
<td>1,6781</td>
</tr>
<tr>
<td>40°</td>
<td>339</td>
<td>49,2</td>
<td>1,7573</td>
<td>133</td>
<td>27,4</td>
<td>1,7572</td>
</tr>
<tr>
<td>50°</td>
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<tr>
<td>70°</td>
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<td>1,7624</td>
<td>237</td>
<td>31,4</td>
<td>1,7626</td>
</tr>
</tbody>
</table>

Concluding remarks:

• use of interior-point methods leads to efficient algorithms in shakedown analysis

• choice of starting-point has strong influence on efficiency

• proposed starting-point strategy leads to reduction of CPU-time

Perspectives:

• extension to larger classes of materials: SMA

• industrial application