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Shakedown Analysis Combined with the Problem of Heat Conduction

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Schematic illustration of different material behaviors under varying thermo-mechanical loading.
Schematic illustration of different material behaviors under varying thermo-mechanical loading

- ratcheting
- alternating plasticity
- shakedown
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LOSER-BOUND SHAKEDOWN ANALYSIS

Statical shakedown theorem by Melan*:

* If there exists a loading factor $\alpha > 1$ and a time-independent residual stress field $\bar{\rho}$ such that the yield condition $F \leq 0$ is satisfied for all loads contained within the loading domain $\Omega$ at any time $t$ and at all points $x \in V$ in the volume $V$ of the considered structure, then the system will shake down.

$$ F \left( \alpha \sigma^E(x,t) + \bar{\rho}(x), \sigma_Y(x) \right) \leq 0 $$

where:

- $\sigma_Y(x)$: yield stress
- $\sigma^E(x,t)$: elastic reference stress

Mathematical formulation of the statical shakedown theorem as an optimization problem:

\[
\begin{align*}
\max & \quad \alpha \\
\text{s.t.} & \quad \sum_{r=1}^{NG} C_r \cdot \bar{\rho}_r = 0 \\
& \quad \forall r \in [1, NG], \forall j \in [1, NC]: \quad F(\alpha \sigma_{r,j} + \bar{\rho}_r, \sigma_{y,r}) \leq 0
\end{align*}
\]

where:
- \( r \): considered Gaussian point
- \( NG \): total number of Gaussian points
- \( j \): considered corner of the loading domain \( \Omega \)
- \( NC \): total number of corners of the loading domain
- \( C_r \): equilibrium matrixes which guarantee that \( \bar{\rho}_r \) is self-equilibrated
- \( \bar{\rho}_r \): affine linear equality constraints
- \( F(\cdot) \): nonlinear, convex inequality constraints
- \( \alpha \): linear objective function (convex)
**SOLUTION BY INTERIOR-POINT METHOD**

\[ \min f(x) = -\alpha \]
\[ A \cdot x = 0 \]
\[ c(x) \geq 0 \]
\[ x \in \mathbb{R}^n \]

introduce slack variables \( w \) and split variables \( y \) and \( z \)

\[ \min f(x) \]
\[ A \cdot x = 0 \]
\[ c(x) - w = 0 \]
\[ x - y + z = 0 \]
\[ w \geq 0, y \geq 0, z \geq 0 \]

introduce barrier parameter \( \mu \)

improved formulation:

\[ \nu. \text{ Mises yield criterion:} \]
\[ c_{r,j}(x) = 2\sigma_{Y,r}^2 - \|u_r - \alpha a_r\|_2^2 \geq 0 \]

solution vector:
\[ x = (u_r \nu \alpha)^T \in \mathbb{R}^n \]
\[ n = 6*NG + 1 \]

\[ \min f_\mu(x, y, z, w) \]
\[ A \cdot x = 0 \]
\[ c(x) - w = 0 \]
\[ x - y + z = 0 \]
\[ w > 0, y > 0, z > 0 \]

where: \[ f_\mu(x) = f(x) - \mu \left[ \sum \log w_j + \sum \log y_i + \sum \log z_i \right] \]
Karush-Kuhn-Tucker (KKT) conditions:

solution is optimal if the Lagrangian $L$ of the problem possesses a saddle point (necessary and sufficient condition for convex problems)

Lagrangian:  
\[ L = f_{\mu}(x, y, z, w) - \lambda_E \cdot (A \cdot x) - \lambda_I \cdot (c(x) - w) - s \cdot (x - y + z) \]

with Lagrange multipliers:  
\[ \lambda_E \in \mathbb{R}^{m_E}, \lambda_I \in \mathbb{R}^{m_I}, s \in \mathbb{R}^n \]

Saddle point condition:  
\[ \nabla L = 0 \quad \text{in each direction} \]

System of nonlinear equations approximately solved by Newton’s method
Newton’s method:

New variables $\Pi_{k+1}$ of the subsequent iteration step $k+1$ are computed from the variables $\Pi_k$ of the previous one $k$:

$$\Pi_{k+1} = \Pi_k + \Upsilon_k \cdot \Delta \Pi_k$$

Step values $\Delta \Pi_k$ are the solution of the linearized system:

$$J(\Pi_k) \cdot \Delta \Pi_k = -\nabla L(\Pi_k)$$

where: $J(\Pi_k)$: Jacobian of $\nabla L(\Pi)$

$\Upsilon_k$: diagonal matrix of damping factors
SOLUTION BY INTERIOR-POINT METHOD

Initialization:
- elastic stresses
- C-matrix
- transformation

Initialization

- Update variables

- Update KKT

- Damp with Non-negativity

- Damp with Linesearch

Update barrier

- NO

- Break condition, outer iter.

- NO

- YES

Update variables

- YES

- Break condition, inner iter.

- NO

- YES

Done

- YES
NUMERICAL EXAMPLE

Plate with six wholes under variable temperature loading:

Material: 2024-T6 aluminum, all material parameters are considered as temperature-independent

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height $h$ in [mm]</td>
<td>800</td>
</tr>
<tr>
<td>Width $b$ in [mm]</td>
<td>400</td>
</tr>
<tr>
<td>Thickness $t$ in [mm]</td>
<td>10</td>
</tr>
<tr>
<td>Diameter $d$ in [mm]</td>
<td>100</td>
</tr>
<tr>
<td>plate initial temperature in [K]</td>
<td>300</td>
</tr>
<tr>
<td>surrounding temperature in [K]</td>
<td>300</td>
</tr>
<tr>
<td>maximum temperature load in [K]</td>
<td>500</td>
</tr>
</tbody>
</table>
Elementation of the structure:

Element-type: square, 8 nodes per element

solid45, solid70

Number of nodes: 1798

Number of elements: 768

Loading cases: 2
Temperature fields in the two loading cases:

Computed by ANSYS with element *solid70*:
NUMERICAL EXAMPLE

Equivalent von Mises stresses in the two loading cases:

Computed by ANSYS with element *solid45*:
NUMERICAL EXAMPLE

Results of the shakedown analysis:

EL : Elastic Limit
SD: Shakedown
AP: Alternating Plasticity
NUMERICAL EXAMPLES

Square plate with a circular hole with three-dimensional loading space:

All three loads $P_x$, $P_y$ and $T$ vary independently:

\[
0 \leq P_x \leq \mu_x P_0 \\
0 \leq P_y \leq \mu_y P_0 \\
0 \leq T \leq \mu_T T_0
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length $L$ [mm]</td>
<td>100</td>
</tr>
<tr>
<td>Thickness $t$ [mm]</td>
<td>2</td>
</tr>
<tr>
<td>Diameter $D$ [mm]</td>
<td>20</td>
</tr>
</tbody>
</table>

All material parameters are considered as temperature-independent.
NUMERICAL EXAMPLES

FE-mesh and relevant numbers of optimization problem:

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elements $NE$</td>
<td>400</td>
</tr>
<tr>
<td>Gaussian points $NG$</td>
<td>3200</td>
</tr>
<tr>
<td>Corners $NC$</td>
<td>8</td>
</tr>
<tr>
<td>Variables</td>
<td>131201</td>
</tr>
<tr>
<td>Equality constraints $m_E$</td>
<td>114646</td>
</tr>
<tr>
<td>Inequality constraints $m_I$</td>
<td>25600</td>
</tr>
</tbody>
</table>
NUMERICAL EXAMPLES

Result in three-dimensional loading space:
CONCLUSIONS

Concluding remarks:

• use of interior-point methods leads to efficient algorithms in shakedown analysis
• improved formulation has been presented for v.Mises yield criterion
• method allows calculation of systems with multidimensional loading

Perspectives:

• extension to larger classes of materials: e.g. limited kinematical hardening
• numerical advancements: selective algorithm
• industrial application