Nonlinear Elastic Waves in a 1D Layered Composite Material: Some Numerical Results

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Overview

- Introduction
- Given problem:
  - Propagation of a long wave through a 1D composite
  - Periodically repeated structure of the composite: Matrix with inclusions
  - Nonlinear elastic behavior of the components
- Application of the asymptotic homogenization method (AHM)
  - Homogenized wave equation of different orders
- Stationary strain waves
- Example
- Conclusion
Introduction
Motivation

- Increasing number of applications for composite materials
  - civil engineering
  - mechanical engineering
  - biomechanics
  - ...

- Challenge: identification of the internal structure of heterogeneous solids by means of measurements of macroscopic properties
  - non-destructive testing of composite materials
  - non-invasive physical examination (med.)
  - detection of the texture of soils and rocks (geo science)
  - ...

- Popular model: The heterogeneous material is replaced by a homogeneous one with the same mechanical properties
Governing Relations
Governing Relations

- One-dimensional wave propagation in $x_1 = x$-direction
- Periodically repeated structure of the composite:
  - $\Omega^{(1)}$: matrix
  - $\Omega^{(2)}$: inclusion
- Nonlinear Problem
Governing Relations

Physical nonlinearity

Murnaghan potential\(^1\): Nonlinear elastic behavior of the component \(\Omega^{(k)}, k = 1, 2\):

\[
\Phi^{(k)} = \left\{ \begin{array}{l}
\text{linear properties} \\
\frac{1}{2} \lambda^{(k)} \left( \varepsilon^{(k)}_{nn} \right)^2 + \mu^{(k)} \left( \varepsilon^{(k)}_{ij} \right)^2 \\
+ \frac{1}{3} A^{(k)} \varepsilon^{(k)}_{ij} \varepsilon^{(k)}_{in} \varepsilon^{(k)}_{jn} + B^{(k)} \left( \varepsilon^{(k)}_{ij} \right)^2 \varepsilon_{nn} + \frac{1}{3} C^{(k)} \left( \varepsilon^{(k)}_{nn} \right)^3, \\
\end{array} \right. \\
\right. \\
\text{physical nonlinearity}
\]

\(\lambda^{(k)}, \mu^{(k)}\): Lamé parameters
\(A^{(k)}, B^{(k)}, C^{(k)}\): Landau constants\(^2\)

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\(^1\)Murnaghan, F. D. 1951 *Finite deformation of an elastic solid*. NY: Wiley.

Geometrical nonlinearity

Cauchy-Green tensor:
Relation between strain $\varepsilon_{ij}^{(k)}$ and displacement $u_i^{(k)}$

$$\varepsilon_{ij}^{(k)} = \frac{1}{2} \left( \frac{\partial u_i^{(k)}}{\partial x_j} + \frac{\partial u_j^{(k)}}{\partial x_i} + \frac{\partial u_n^{(k)}}{\partial x_j} \frac{\partial u_n^{(k)}}{\partial x_i} \right)$$
The 1D wave equation for component $\Omega^{(k)}, \ k = 1, 2$ is\(^3\):

$$\alpha^{(k)} \frac{\partial^2 u^{(k)}}{\partial x^2} + \beta^{(k)} \frac{\partial u^{(k)}}{\partial x} \frac{\partial^2 u^{(k)}}{\partial x^2} = \rho^{(k)} \frac{\partial^2 u^{(k)}}{\partial t^2},$$

where

$$\alpha^{(k)} = \lambda^{(k)} + 2\mu^{(k)},$$
$$\beta^{(k)} = 3 \left[ (\lambda^{(k)} + 2\mu^{(k)}) + 2 \left( A^{(k)} + 3B^{(k)} + C^{(k)} \right) \right],$$
$$\rho^{(k)} : \text{mass density},$$
$$u^{(k)} : \text{longitudinal displacement},$$

Governing Relations

Longitudinal stresses:

$$\sigma^{(k)} = \alpha^{(k)} \frac{\partial u^{(k)}}{\partial x} + \frac{\beta^{(k)}}{2} \left( \frac{\partial u^{(k)}}{\partial x} \right)^2$$

Longitudinal stresses on the interface $\partial \Omega$ are equal:

$$\left\{ \alpha^{(1)} \frac{\partial u^{(1)}}{\partial x} + \frac{\beta^{(1)}}{2} \left( \frac{\partial u^{(1)}}{\partial x} \right)^2 = \alpha^{(2)} \frac{\partial u^{(2)}}{\partial x} + \frac{\beta^{(2)}}{2} \left( \frac{\partial u^{(2)}}{\partial x} \right)^2 \right\} \bigg|_{\partial \Omega}$$

Perfect bonding on the interface $\partial \Omega$:

$$\left\{ u^{(1)} = u^{(2)} \right\} \bigg|_{\partial \Omega}$$
Asymptotic homogenization method
Asymptotic homogenization method

We consider:

- wave of the wavelength $L$ travelling through the composite
- length $\ell$ is comparable to the size of the heterogeneities
- $\ell \ll L$
- small parameter $L = \varepsilon^{-1}\ell$.

We introduce:

- fast coordinate variable $y$, measuring the displacement within the periodically repeated unit cell
- slow coordinate variable $x$, measuring the displacement in the area of interest
- $y = \varepsilon^{-1}x$
Asymptotic homogenization method

The longitudinal displacement is now introduced as an asymptotic expansion:

\[ u^{(k)} = u_0(t, x) + \epsilon u_1^{(k)}(t, x, y) + \epsilon^2 u_2^{(k)}(t, x, y) + \ldots \]

with

- \( u_0 = u_0(t, x) \): Homogenized term depending only on the slow coordinate and time
- \( u_i^{(k)} = u_i^{(k)}(t, x, y) \): Term providing corrections on the order of \( \epsilon^i \). Depends on both fast and slow coordinates.
The spatial periodicity of the medium results in:

$$u_i^{(k)}(t, x, y) = u_i^{(k)}(t, x, y + L)$$

The derivatives with respect to fast and slow variables are

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \varepsilon^{-1} \frac{\partial}{\partial y}$$

$$\frac{\partial^2}{\partial x^2} \rightarrow \frac{\partial^2}{\partial x^2} + 2\varepsilon^{-1} \frac{\partial^2}{\partial x \partial y} + \varepsilon^{-2} \frac{\partial^2}{\partial y^2}$$
Asymptotic homogenization method

Bonding condition with respect to fast and slow variables:

\[
\{ u^{(2)} = u^{(1)} \} \bigg|_{\partial \Omega} \Rightarrow \{ u^{(2)}_i = u^{(1)}_i \} \bigg|_{\partial \Omega}
\]

Equality of stresses at the boundary with respect to fast and slow variables:

\[
\begin{align*}
\left\{ \alpha^{(1)} \frac{\partial u^{(1)}}{\partial x} + \frac{\beta^{(1)}}{2} \left( \frac{\partial u^{(1)}}{\partial x} \right)^2 = \alpha^{(2)} \frac{\partial u^{(2)}}{\partial x} + \frac{\beta^{(2)}}{2} \left( \frac{\partial u^{(2)}}{\partial x} \right)^2 \right\} \bigg|_{\partial \Omega} \\
\downarrow \downarrow \downarrow \downarrow
\end{align*}
\]

\[
\begin{align*}
\left\{ \alpha^{(1)} \left( \frac{\partial u^{(1)}_{i-1}}{\partial x} + \frac{\partial u^{(1)}_i}{\partial y} \right) + \frac{\beta^{(1)}}{2} \sum_{m=0}^{i-1} \left( \frac{\partial u^{(1)}_m}{\partial x} + \frac{\partial u^{(1)}_{m+1}}{\partial y} \right) \left( \frac{\partial u^{(1)}_{i-m-1}}{\partial x} + \frac{\partial u^{(1)}_{i-m}}{\partial y} \right) \right. \\
= \alpha^{(2)} \left( \frac{\partial u^{(2)}_{i-1}}{\partial x} + \frac{\partial u^{(2)}_i}{\partial y} \right) + \frac{\beta^{(2)}}{2} \sum_{m=0}^{i-1} \left( \frac{\partial u^{(2)}_m}{\partial x} + \frac{\partial u^{(2)}_{m+1}}{\partial y} \right) \left( \frac{\partial u^{(2)}_{i-m-1}}{\partial x} + \frac{\partial u^{(2)}_{i-m}}{\partial y} \right) \right\} \bigg|_{\partial \Omega}
\end{align*}
\]
Asymptotic homogenization method

The macroscopic wave equation becomes

\[
\bar{\alpha} \frac{\partial^2 u}{\partial x^2} + \bar{\beta} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \bar{\alpha} \gamma L^2 \varepsilon^2 \frac{\partial^4 u}{\partial x^4} = \bar{\rho} \frac{\partial^2 u}{\partial t^2}
\]

where

\[
\bar{\alpha} = \frac{\alpha^{(1)} \alpha^{(2)}}{c^{(1)} \alpha^{(2)} + c^{(2)} \alpha^{(1)}}, \quad \bar{\beta} = \frac{c^{(1)} \beta^{(1)} (\alpha^{(2)})^3 + c^{(2)} \beta^{(2)} (\alpha^{(1)})^3}{(c^{(1)} \alpha^{(2)} + c^{(2)} \alpha^{(1)})^3},
\]

\[
\bar{\rho} = c^{(1)} \rho^{(1)} + c^{(2)} \rho^{(2)}, \quad \bar{\gamma} = \frac{(c^{(1)} c^{(2)})^2}{12} \frac{\nu_0^4}{(\nu^{(1)} \nu^{(2)})^2} \left( \frac{z^{(1)}}{z^{(2)}} - \frac{z^{(2)}}{z^{(1)}} \right)^2
\]

\( c^{(n)} \): volume fractions of the components,

\( \nu_0 = \sqrt{\bar{\alpha}/\bar{\rho}} \): wave velocity at \( \bar{\beta} = 0 \) and \( \varepsilon = 0 \)

\( \nu^{(n)} = \sqrt{\alpha^{(n)}/\rho^{(n)}} \): wave velocity of component \( n \),

\( z^{(n)} = \sqrt{\alpha^{(n)} \rho^{(n)}} \): acoustic impedance of component \( n \)

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Stationary Strain Waves

Stationary solution of

\[
\bar{\alpha} \frac{\partial^2 u}{\partial x^2} + \bar{\beta} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \bar{\alpha} \bar{\gamma} L^2 \varepsilon^2 \frac{\partial^4 u}{\partial x^4} = \bar{\rho} \frac{\partial^2 u}{\partial t^2}
\]
Stationary Strain Waves

Introduction of
\[ z = x - \nu t \]

\[ \begin{align*}
\text{displacement} & \quad u(x, t) = u(z) \\
\text{strain} & \quad f = \frac{du}{dz}
\end{align*} \]

\[ \bar{\alpha} \frac{\partial^2 u}{\partial x^2} + \bar{\beta} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \bar{\alpha} \bar{\gamma} L^2 \varepsilon^2 \frac{\partial^4 u}{\partial x^4} = \bar{\rho} \frac{\partial^2 u}{\partial t^2} \]

\[ \frac{d^2 f}{dz^2} + af + bf^2 + c = 0 \]

with

\[ a = \left(1 - \nu^2 / \nu_0^2\right) / (\bar{\gamma} l^2), \quad b = \bar{\beta} / (2 \bar{\alpha} \bar{\gamma} l^2), \quad c : \text{integration constant} \]
The here presented problem is now analysed for:

- initial conditions $f(0) = -\frac{A_0}{2}$ and $\frac{df(0)}{dz} = 0$
- weak nonlinearity
- strong nonlinearity
Weak nonlinearity:
Asymptotic solution by the Lindstedt-Pointcaré technique

Strain:

\[ f = -A_0 \left[ \left( \frac{1}{2} + \frac{1}{24} \eta + \frac{5}{1152} \eta^2 + \frac{5}{13824} \eta^3 + \frac{1}{221184} \eta^4 \right) \cos(kz) \right. \]
\[ \left. - \left( \frac{1}{24} \eta + \frac{1}{144} \eta^2 + \frac{1}{1152} \eta^3 + \frac{1}{13824} \eta^4 \right) \cos(2kz) \right. \]
\[ \left. + \left( \frac{1}{384} \eta^2 + \frac{1}{1536} \eta^3 + \frac{1}{9216} \eta^4 \right) \cos(3kz) \right. \]
\[ \left. - \left( \frac{1}{6912} \eta^3 + \frac{1}{20736} \eta^4 \right) \cos(4kz) \right. \]
\[ \left. + \frac{5}{663552} \eta^4 \cos(5kz) \right) + O(\eta^5) \]

Phase velocity:

\[ \frac{v^2}{v_0^2} = 1 - \bar{\gamma} k^2 \ell^2 \left[ 1 - \frac{1}{24} \eta^2 - \frac{1}{144} \eta^3 - \frac{5}{4608} \eta^4 + O(\eta^5) \right] \]

Wave number:

\[ k^2 = a \left[ 1 + \frac{1}{24} \zeta^2 + \frac{1}{144} \zeta^3 - \frac{1}{1536} \zeta^4 + O(\eta^5) \right] \]

where \( \eta = -bA_0/k^2 \) and \( \zeta = -bA_0/a \).

Stationary Strain Waves

Comparison of the asymptotic solution (dashed line) with the exact one (solid line)

a) $s = 0.9$

b) $s = 0.94$
Stationary Strain Waves

Strong nonlinearity: Exact integration in elliptic functions

Strain: \[ f = -\frac{A_0}{2} + \frac{A_0 s^2}{2[1 - E(s)/K(s)]} \text{sn}^2 \left( \frac{k_0}{2} z, s \right) \]

Phase velocity: \[ \frac{\nu^2}{\nu_0^2} = 1 - \gamma k_0^2 \ell^2 \left[ 3E(s)/K(s) - 2 + s^2 \right] \]

Wave number: \[ k = \frac{\pi}{2K(s)} k_0, \quad L = \frac{2\pi}{k} = \frac{4K(s)}{k_0} \]

where

- \( \text{sn}(\cdot) \): elliptic sine,
- \( K(s), E(s) \) complete elliptic integrals of the 1st and 2nd kind

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Stationary Strain Waves

$s\, (0 \leq s \leq 1)$: modulus of the elliptic functions/nonlinear factor. It can be calculated from the transcendental equation:

$$2[1 - E(s)/K(s)] = -bA_0/k_0^2$$

$s \to 0$
(purely harmonic wave)

\[
\begin{align*}
    f &= \frac{A_0}{2} \cos(kz), \\
    u &= \frac{A_0}{2k} \sin(kz), \\
    \nu^2/\nu_0^2 &= 1 - \bar{\gamma}k^2\ell^2,
\end{align*}
\]

$s \to 1$
(solitary strain wave, shock disp. wave)

\[
\begin{align*}
    f &= \frac{A_0}{2} \text{sech}^2(z/h), \\
    u &= \frac{A_0h}{2} \tanh(z/h), \\
    \nu^2/\nu_0^2 &= 1 + 4\bar{\gamma}(\ell/h)^2
\end{align*}
\]

where $h = 2/k_0$ and $k_0 = 2kK(s)/\pi$

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Stationary Strain Waves

Comparison of the asymptotic solution (dashed line) with the exact one (solid line)

a) $s = 0.9$

b) $s = 0.94$
Example:
Steel-aluminium composite
\[
\bar{\alpha} \frac{\partial^2 u}{\partial x^2} + \bar{\beta} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \alpha \gamma L^2 \varepsilon^2 \frac{\partial^4 u}{\partial x^4} = \bar{\rho} \frac{\partial^2 u}{\partial t^2}
\]

- \( \ell = 10^{-3} \text{ m}, \quad \ell^{(1)} = \ell^{(2)} = 0.5 \ell \)
- \( \bar{\alpha} = 160 \text{ GPa}, \quad \bar{\beta} = -2385 \text{ GPa}, \quad \bar{\gamma} = 0.0171, \quad \bar{\rho} = 5250 \text{ kg/m}^3 \)
Nonlinear factor $s$ as a function of the strain amplitude $A_0$ and the frequency $\omega$. 
Phase velocity $\nu$ at different values of the nonlinear factor $s$ and of the dispersion parameter $\varepsilon = \ell/L = kl/(2\pi)$.

$\nu$ exceeds $\nu_0 = \sqrt{\tilde{\alpha}/\tilde{\rho}}$ and the wave passes from a subsonic to a supersonic mode at $s = 0.9804$.
Dispersion curves. Horizontal dashed lines correspond to the first SB threshold determined from the condition $d\omega/dk = 0$. Here $\omega_0 = k\nu_0$ is the frequency in the quasi-homogeneous limit $\varepsilon = 0$. 
Example

Influence of the nonlinear factor $s$ on the frequency $\omega_{SB}$ of the first stop band.
Conclusion
Conclusion

In this work nonlinear waves are investigated. Wave characteristics:

- shape of the wave
- velocity
- attenuation/dispersion

depending on

- amplitude of the signal
- parameters of the structure
Conclusion

Open problems:

- study of nonlinear waves in 2D and 3D problems
- examination of nonlinear vibration of heterogeneous structures of finite sizes
Thank you for your attention